The solution of Eq. (12) with initial and boundary conditions defined by Eqs. (4–6) is

$$\frac{T - T_0}{T_m - T_0} = \frac{1 - \exp\left(\dfrac{-v\xi}{\alpha_c}\right)}{1 - \exp\left(\dfrac{-v\delta_c}{\alpha_c}\right)} \tag{14}$$

which is the temperature distribution in the gas-char layer.

Differentiating Eq. (14) with respect to ξ, substituting into Eqs. (9) and (13), and simplifying give

$$v = \frac{h(T_g - T_0)}{\rho_s \Delta H_{\text{eff}}} \tag{15}$$

$$\delta_c = \frac{\alpha_c}{v} \ln \frac{\Delta H_{\text{eff}}}{L} \tag{16}$$

where

$$\Delta H_{\text{eff}} = L + c(T_0 - T_m)\rho_c/\rho_s \tag{17}$$

This ΔH_{eff} is called effective heat of ablation, which differs from L, the heat of charring. It is not the value of heat of reaction, which has been neglected in this derivation.

Virgin material layer

The solution of Eq. (1) under steady state with the boundary conditions (7) and (8) can be easily derived as

$$\frac{T - T_i}{T_m - T_i} = \frac{x - \delta_s}{\delta_c - \delta_s} \tag{18}$$

Transient Solution

Char-gas layer

The governing equations are Eqs. (3–6, 9, and 11). The method of solution is similar to that of the melting problem described by Carslaw and Jaeger[1] which has been applied to ablation by Grosh.[2] The method is as follows. Let

$$s = 2\lambda(\alpha\theta)^{1/2} \tag{19}$$

which means that the moving boundary s is proportional to the square root of the product of the thermal diffusivity and time. λ is a proportional constant that will be determined from the environmental condition and will be described later. Then the ablation velocity is the derivative of Eq. (19) with respect to time or

$$ds/d\theta = \lambda(\alpha/\theta)^{1/2} \tag{20}$$

It has been verified that one of the solutions of Eq. (3) is

$$T = B_1 + B_2 \operatorname{erf}\left(\frac{\xi}{2(\alpha\theta)^{1/2}} + \lambda\right) \tag{21}$$

The constants B_1 and B_2 are determined from boundary conditions (5) and (6). The solution is

$$\frac{T - T_0}{T_m - T_0} = \frac{\operatorname{erf}[(\xi/2(\alpha\theta)^{1/2}) + \lambda] - \operatorname{erf}\lambda}{\operatorname{erf}2\lambda - \operatorname{erf}\lambda} \tag{22}$$

Differentiating Eq. (22) with respect to ξ, substituting into Eqs. (9) and (11), and simplifying give

$$\frac{T_0 - T_m}{(\pi\alpha\theta)^{1/2}[\operatorname{erf}2\lambda - \operatorname{erf}\lambda]\exp\lambda^2} = \frac{h}{k}(T_g - T_0) \tag{23}$$

$$c(T_0 - T_m)/L(\pi)^{1/2} = \lambda[\operatorname{erf}2\lambda - \operatorname{erf}\lambda]\exp 4\lambda^2 \tag{24}$$

With these two equations, the unknowns T_0 and λ can be solved. Then the char depth is given by Eq. (19) and the ablation velocity by Eq. (20). The amount of material blown away is the area under the curve of ablation velocity vs time.

Virgin material layer

The governing equations are Eqs. (1, 7, and 8). As before, one of the solutions is

$$T_s = B_3 + B_4 \operatorname{erf}[x/2(\alpha_s\theta)^{1/2}] \tag{25}$$

The constants B_3 and B_4 can be determined from boundary conditions and Eqs. (7) and (8). The final result is

$$\frac{T_s - T_m}{T_i - T_m} = \frac{\operatorname{erf}[x/2(\alpha_s\theta)^{1/2}] - \operatorname{erf}[\lambda(\alpha_c/\alpha_s)^{1/2}]}{\operatorname{erf}[\delta_s/2(\alpha_s\theta)^{1/2}] - \operatorname{erf}[\lambda(\alpha_c/\alpha_s)^{1/2}]} \tag{26}$$

Discussion

If the ablative material is porous, the thermal diffusivity in the char-gas layer will be replaced by an effective thermal diffusivity defined by

$$\alpha_{\text{eff}} = k_{\text{eff}}/(\rho c)_{\text{eff}} \tag{27}$$

where

$$k_{\text{eff}} = k_g\epsilon + k_c(1 - \epsilon) \tag{28}$$

$$(\rho c)_{\text{eff}} = \epsilon\rho_g c_g + (1 - \epsilon)\rho_c c_c \tag{29}$$

Although these equations are developed on the basis of negligible transpiration and diffusion effects, they still can be applicable to take these effects into account by adjusting the heat-transfer coefficient h

References

[1] Carslaw, H. S. and Jaeger, J. C., *Conductions of Heat in Solids* (Oxford University Press, New York, 1959), 2nd ed., pp. 285–286, 315–316.

[2] Grosh, R. J., "Transient temperature in a semi-infinite porous solid with phase change and transpiration effects," Wright Air Development Div. TR 60-105, Midwest Applied Science Corp. Lafayette, Ind. (January 1960).

Closed-Form Lagrangian Multipliers for Coast Periods of Optimum Trajectories

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Nomenclature

r = radial distance from center of reference body
v = velocity
γ = complement of the flight-path angle
F = thrust
α = angle of attack
m = mass
g = acceleration due to gravity
λ_i = Lagrangian multipliers

1. Introduction

APPLICATION of the classical calculus of variations to trajectory optimization problems results, in general, in a system of simultaneous differential equations for the Lagrangian multipliers (essentially the direction cosines of the thrust vector) which, because of the nonexistence of or difficulty of obtaining an analytical solution, are integrated numerically to obtain values that determine the optimum

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trajectory for specified initial conditions. In many engineering applications, it is necessary to simulate vacuum coast phases within optimum trajectories in a central force gravity field; during these phases it is necessary to continue to evaluate the Lagrangian multipliers to obtain their values at engine restart, even though it is evident that no control over the trajectory exists when thrust is zero. The determination of a closed-form solution for these multipliers and the equations of motion would give benefits of both increased accuracy and savings in computation time by reducing the calculations necessary on a coast arc to those required to determine two points, namely, that at the beginning of coast and that at the time of engine restart. It is shown in this paper that the equations of motion and the Lagrangian multiplier rates can be integrated analytically to determine a closed-form time-dependent representation of a trajectory at any point on a coast arc in a vacuum in a central-force gravity field.

Phases of trajectory optimization problems currently under study involving the simulation of portions of trajectories during which the vehicle is coasting and for which the optimum (in the sense of minimizing velocity losses due to gravity) thrust-vector orientation must be determined include: injection into and out of parking orbits; the problem, discussed in Ref. 1, of determining the optimum time to resume thrusting to leave a given orbit; and the simulation of interplanetary transfer orbits.

The solution formulated in this paper provides closed-form equations for the resumption of thrust at any point of a Keplerian orbit. Thus, it is possible to eliminate time-consuming numerical integration techniques otherwise used. Combining these results with the well-known Keplerian equations of motion, it is possible to obtain a closed-form time-dependent representation of the trajectory and optimizing functions at any point on a coast arc.

2. Statement of the Problem

Given the equations of state,

$$\dot{x}_i = f_i(x_1, \ldots, x_r, \alpha, t) \qquad 1 \leq i \leq n, 1 \leq r \leq n \quad (2.1)$$

associated with trajectories assumed to be characterized by the following properties: two-dimensional, point mass vehicle, vacuum, nonrotating reference body, central-force gravity field, constant thrust along the vehicle centerline; and given the Euler-Lagrange equations

$$\dot{\lambda}_i = -\partial H/\partial x_i \qquad 1 \leq i \leq n \quad (2.2)$$

$$\partial H/\partial \alpha = 0 \quad (2.3)$$

where H is the Hamiltonian

$$H = \sum_{i=1}^{n} \lambda_i f_i \quad (2.4)$$

determine, for trajectory arcs for which

$$F = 0 \quad (2.5)$$

a closed-form representation

$$\lambda_i = \lambda_i(x_1, \ldots, x_s, t) \quad 1 \leq i \leq n \qquad s \leq n$$

3. Equations of State and Variational Equations

The equations of state appropriate to the problem are taken to be

$$\begin{aligned} \dot{m} &= f_1 = \text{const} \\ \dot{r} &= f_2 = v \cos\gamma \\ \dot{v} &= f_3 = F \cos\alpha/m - g \cos\gamma \\ \dot{\gamma} &= f_4 = F \sin\alpha/mv + (g/v - v/r) \sin\gamma \end{aligned}$$

where

$$g = GM/r^2 \qquad GM = \text{const}$$

The state variables are related as indicated by Fig. 1. With the f_i as defined previously, the Lagrangian multiplier rates, by (2.2), are

$$\begin{aligned} \dot{\lambda}_1 &= \lambda_3(F \cos\alpha/m^2) + \lambda_4(F \sin\alpha/m^2 v) \\ \dot{\lambda}_2 &= -\lambda_3(2g/r) \cos\gamma + \lambda_4(2g/rv - v/r^2) \sin\gamma \\ \dot{\lambda}_3 &= -\lambda_2 \cos\gamma + \lambda_4[(g/v^2 + 1/r) \sin\gamma + F \sin\alpha/mv^2] \\ \dot{\lambda}_4 &= \lambda_2 v \sin\gamma - \lambda_3 g \sin\gamma + \lambda_4(v/r - g/v) \cos\gamma \end{aligned}$$

By Eq. (2.3),

$$\alpha = \tan^{-1}(\lambda_4/v\lambda_3)$$

Since H is explicitly independent of time, the first integral of the Euler-Lagrange equations yields H = const. The f_i introduced previously assume $\dot{m} \geq 0$, and so Eq. (2.4) is

$$\lambda_1(-\dot{m}) + \lambda_2\dot{r} + \lambda_3\dot{v} + \lambda_4\dot{\gamma} = \text{const}$$

Introducing the assumption of zero thrust, the equations of state become

$$\dot{m} = 0 \quad (3.1)$$

$$\dot{r} = v \cos\gamma \quad (3.2)$$

$$\dot{v} = -g \cos\gamma \quad (3.3)$$

$$\dot{\gamma} = (g/v - v/r) \sin\gamma \quad (3.4)$$

with

$$g = GM/r^2 \quad (3.5)$$

and the variational equations are

$$\dot{\lambda}_1 = 0 \quad (3.6)$$

$$\dot{\lambda}_2 = -\lambda_3(2g/r) \cos\gamma + \lambda_4(2g/rv - v/r^2) \sin\gamma \quad (3.7)$$

$$\dot{\lambda}_3 = -\lambda_2 \cos\gamma + \lambda_4(g/v^2 + 1/r) \sin\gamma \quad (3.8)$$

$$\dot{\lambda}_4 = \lambda_2 v \sin\gamma - \lambda_3 g \sin\gamma + \lambda_4(v/r - g/v) \cos\gamma \quad (3.9)$$

$$\lambda_2\dot{r} + \lambda_3\dot{v} + \lambda_4\dot{\gamma} = C_0 \quad (3.10)$$

where C_0 is a constant. (Hereafter, C_i and K_i terms will always represent constants.)

4. Closed-Form Lagrangian Multipliers

Integrals of the state Eqs. (3.3) and (3.4) readily can be verified to be

$$v^2 = (2GM/r) + 2C_1 \quad (4.1)$$

$$\sin\gamma = C_2/vr \quad (4.2)$$

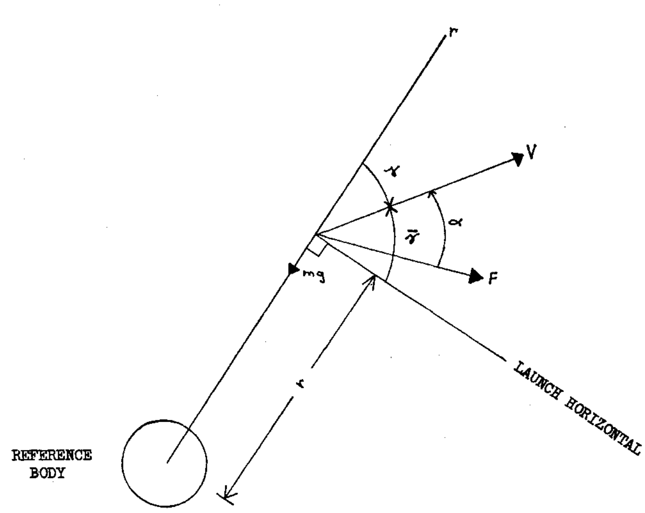

Fig. 1 Point mass vehicle in flight path coordinates.

These and the equations of Sec. 3 are used to integrate the variational equations for the two cases of noncircular and circular orbits as follows.

A. Noncircular orbits

Multiply Eq. (3.9) by $\cot\gamma$ and obtain, using Eqs. (3.2) and (3.3),

$$\dot{\lambda}_4 \cot\gamma = \lambda_2 \dot{r} + \lambda_3 \dot{v} - \lambda_4 \dot{\gamma} \cot^2\gamma$$

Substitute from Eq. (3.10) to obtain

$$\dot{\lambda}_4 \cot\gamma + \lambda_4 \dot{\gamma} \csc^2\gamma = C_0 \tag{4.3}$$

From Eqs. (4.2, 3.4, 3.5, and 3.2),

$$\cos\gamma = [1 - C_2{}^2/v^2r^2]^{1/2}$$

$$\dot{\gamma} \csc^2\gamma = (1/C_2)[(-GM/r) - 2C_1]$$

$$dt = 1/v[1 - C_2{}^2/v^2r^2]^{-1/2}dr$$

Make these substitutions into Eq. (4.3) and use v from Eq. (4.1) to obtain λ_4 as a function of r. The result is

$$\frac{d\lambda_4}{dr} + \lambda_4 \frac{-2C_1r - GM}{2C_1r^2 + 2GMr - C_2{}^2} = C_0C_2 \frac{r}{2C_1r^2 + 2GMr - C_2{}^2} \tag{4.4}$$

This first-order linear differential equation has the solution[2]

$$\lambda_4 = K_0 - K_1GMr + K_2vr\cos\gamma \tag{4.5}$$

where

$$K_1 = -C_0C_2/[2C_2{}^2C_1 + (GM)^2]$$

$$K_0 = K_1C_2{}^2 \qquad K_2 = \text{constant of integration}$$

and we have written

$$[2C_1r^2 + 2GMr - C_2{}^2]^{1/2} = vr\cos\gamma$$

as can be verified using Eqs. (4.2) and (4.1). Equation (4.5) is valid for any noncircular orbit where

$C_1 < 0$ (elliptic orbit)

$C_1 = 0$ (parabolic orbit)

$C_1 > 0$ (hyperbolic orbit)

The integration of $\dot{\lambda}_2$ is accomplished similarly: Substitute λ_3 from (3.10) into (3.7), simplify, and obtain

$$\dot{\lambda}_2 = -2\lambda_2(\dot{r}/r) + \lambda_4(v/r^2)\sin\gamma + 2C_0/r$$

Use (3.1–3.5, 4.1, 4.2, and 4.5) to express this equation in terms of r and λ_2 to obtain

$$\frac{d\lambda_2}{dr} + \lambda_2 \frac{2}{r} = \frac{K_0C_2}{r^2[2C_1r^2 + 2GMr - C_2{}^2]^{1/2}} - \frac{K_1\,GM\,C_2}{r[2C_1r^2 + 2GMr - C_2{}^2]^{1/2}} + \frac{K_2C_2}{r^2} + \frac{2C_0}{[2C_1r^2 + 2GMr - C_2{}^2]^{1/2}}$$

The integral of this equation is dependent in form upon the energy with which the orbit is attained:

For $C_1 < 0$ (elliptic orbit)

$$\lambda_2 = \left[\frac{C_0}{2C_1r} - \frac{3C_0\,GM}{4C_1{}^2r^2} - \frac{K_1\,GM\,C_2}{2C_1r^2}\right] vr\cos\gamma + \left[\frac{K_0C_2}{r^2} + \frac{K_1(GM)^2C_2}{2C_1r^2} + C_0\frac{3(GM)^2 + 2C_1C_2{}^2}{4C_1{}^2r^2}\right] \times$$

[*Equation (4.6a) is continued in the next column.*

$$\frac{\sin^{-1}\{-2C_1r - GM/[(GM)^2 + 2C_1\,C_2{}^2]^{1/2}\}}{[-2C_1]^{1/2}} + \frac{K_2C_2}{r} + \frac{C_{31}}{r^2} \tag{4.6a}$$

For $C_1 > 0$ (hyperbolic orbit)

$$\lambda_2 = \left[\frac{C_0}{2C_1r} - \frac{3C_0\,GM}{4C_1{}^2r^2} - \frac{K_1GMC_2}{2C_1r^2}\right] vr\cos\gamma + \left[\frac{K_0C_2}{r^2} + \frac{K_1(GM)^2C_2}{2C_1r^2} + C_0\frac{3(GM)^2 + 2C_1C_2{}^2}{4C_1{}^2r^2}\right] \times \frac{\log[vr\cos\gamma + r(2C_1)^{1/2} + GM/(2C_1)^{1/2}}{[2C_1]^{1/2}} + \frac{K_2C_2}{r} + \frac{C_{32}}{r} \tag{4.6b}$$

For $C_1 = 0$ (parabolic orbit)

$$\lambda_2 = \frac{1}{r^2}\left\{\left[\frac{K_0C_2}{GM} - \frac{K_1C_2(C_2{}^2 - GMr)}{3GM} + K_2C_2r + C_0\frac{4C_2{}^4 + 4C_2{}^2GMr + 6(GM)^2r^2}{15(GM)^3}\right] \times [2GMr - C_2{}^2]^{1/2} + C_{33}\right\} \tag{4.6c}$$

where the C_{3i} are integration constants. The remaining unknown multiplier can now be obtained from Eq. (3.10):

$$\lambda_3 = (1/\dot{v})(C_0 - \lambda_2\dot{r} - \lambda_4\dot{\gamma}) \tag{4.7}$$

The constants C_i and K_i are determined at the beginning of coast.

B. Circular orbits

For circular orbits, the equations of state are $\dot{r} = 0$, $\dot{v} = 0$, $\dot{\gamma} = 0$, and, since $v \neq 0$, $\gamma = \pi/2$. The variational equations thus become

$$\dot{\lambda}_2 = C_4\lambda_4 \qquad \dot{\lambda}_3 = C_5\lambda_4 \qquad \dot{\lambda}_4 = C_6\lambda_2 + C_7\lambda_3$$

These equations are easily integrated to obtain

$$\lambda_2 = C_8\cos(\beta t + C_9)$$

$$\lambda_3 = C_{10}\cos(\beta t + C_9)$$

$$\lambda_4 = C_{11}\sin\beta t + C_{12}\cos\beta t$$

where $[K]^{1/2} = i\beta$, and $K = C_4C_6 + C_5C_7$, and the constants $C_4 - C_{12}$ are determined by conditions at the beginning of coast.

5. Conclusion

The closed-form Lagrangian multiplier equations (4.5, 4.6a, 4.6b, 4.6c, and 4.7) have been programed on an IBM 7090 computer for comparison with results obtained for the same trajectories from numerical integration of Eqs. (3.7–3.9). The results were in complete agreement within the accuracy limitations (five decimal places) imposed on such simulations.

These results are restricted to a two-dimensional representation; however, an examination of the model for three dimensions indicates that a similar result may be derivable for this case.

References

[1] Leitmann, G., *Optimization Techniques* (Academic Press Inc., New York, 1962), Chap. 4, p. 131.

[2] Sokolnikoff, I. S. and Sokolnikoff, E. S., *Higher Mathematics for Engineers and Physicists* (McGraw-Hill Book Co., Inc., New York, 1941), Chap. 7, p. 284.